DYNAMICS OF BENDING DISTURBANCES OF NONLINEAR VISCOUS LIQUID JETS IN AIR

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The decay of high-velocity laminar jets of extremely viscous liquids is caused by the growth of bending (transverse) disturbances due to the dynamic action of the surrounding air, the pressure of which is greater on concave parts than on convex parts of the jet surface. This type of process has been studied for Newtonian fluid jets [1, 2] by means of the quasi-one-dimensional asymptotic system of equations for the dynamics of liquid jets [1, 3] and by an approximate energy-balance analysis [1]. In this article we generalize the theory of [1, 3] to the case of jets of nonlinear viscous (power-law) liquids whose viscosity depends on the strain rate.

1. We consider the energy balance of a jet of circular cross section in a countercurrent airflow, assuming that its bending under the action of transverse disturbances is planar. We neglect friction forces and drag effects exerted on the jet by the air, so that a plane bending disturbance of the jet axis can be approximately represented by a single harmonic (even in the nonlinear growth stage, as long as the amplitude of the disturbance is not too great:

$$\mathrm{H} = A(t)\sin(\chi s/a_0), \tag{1.1}$$

where χ is the dimensionless wave number ($\chi = 2\pi a_0/l$, l is the wavelength of the disturbance), a_0 is the initial radius of the jet, t is the time, and s is the coordinate measured along the axis of the undisturbed jet.

The work L done in time dt by a linearly distributed aerodynamic "lift" force **q**, applied to a half-wavelength of the disturbed jet and the kinetic energy E of this segment of the jet are given by the relations

$$L = \int\limits_0^{\pi a_0/\chi} q_n \mathrm{H}_{,t}\,ds\,dt, \quad E = \frac{\rho f}{2}\int\limits_0^{\pi a_0/\chi} \mathrm{H}_{,t}^2 \lambda\,ds, \quad \lambda = \sqrt{1+\mathrm{H}_{,s}^2} \tag{1.2}$$

(ρ is the density of the liquid).

The expressions for the projection q_n of the aerodynamic force onto the normal to the jet axis and the cross-sectional area f of the jet have the form [1]

$$q_n = -\rho_1 U_0^2 f \mathrm{H}_{,ss}\left(1+\mathrm{H}_{,s}^2\right)^{-5/2}, \quad f = \pi a^2 = \pi^2 a_0^3\left[\chi \int\limits_0^{\pi a_0/\chi} \lambda\,ds\right], \tag{1.3}$$

where ρ_1 is the density of air and U_0 is the jet velocity.

The extension of the jet in flexure causes a variation of the surface energy $E_1 = \alpha \int\limits_0^{\pi a_0/\chi} 2\pi a \lambda\,ds$ (α is the coefficient of surface tension of the liquid).

Making use of the relation $\partial a^2\lambda/\partial t = 0$, which follows from the incompressibility of the liquid, we find the variation of the surface energy of the jet in time dt:

$$dE_1 = \pi\alpha \int\limits_0^{\pi a_0/\chi} \lambda_{,t} a\,ds\,dt. \tag{1.4}$$

During the growth of the bending disturbances the jet is subjected to uniaxial tension in the first approximation. The rate of relative elongation of a liquid line y = const, z = const parallel to the jet axis (y and z are measured from the center of the cross section

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along the normal $\mathbf{n}$ and the binormal $\mathbf{b}$ to the jet axis; τ is the unit tangent to the axis), i.e., the strain rate, is expressed as follows in the long-wave approximation:

$$D_{\tau\tau} = \frac{[(k^{-1}-y)\,\omega]_{,t}}{(k^{-1}-y)\,\omega} = \lambda^{-1}\lambda_{,t} - ky\frac{\omega_{,t}}{\omega}. \tag{1.5}$$

Here k is the curvature of the jet axis, ω is the angle between the jet cross sections corresponding to the longitudinal coordinates s and s + ds:

$$\omega = -H_{,ss}/(1+H_{,s}^2)\,ds, \tag{1.6}$$

and we have invoked the obvious equality $k^{-1}\omega = \lambda ds$.

The incompressibility condition implies

$$D_{nn} = D_{bb} = -(1/2)D_{\tau\tau}. \tag{1.7}$$

The rheological law for nonlinear viscous liquids is [4]

$$\boldsymbol{\sigma}^* = -p\mathbf{g}^* + 2K[2\mathrm{Sp}(\boldsymbol{D}^{*2})]^{(n-1)/2}\,\boldsymbol{D}^*, \tag{1.8}$$

where $\boldsymbol{\sigma}^*$ and $\boldsymbol{D}^*$ are the stress and strain-rate tensors, $\mathbf{g}^*$ is the metric tensor, p is the pressure, and K and n are the rheological parameters of the liquid. The case of a Newtonian fluid corresponds to K = μ (μ is the viscosity) and n = 1.

Using (1.8), on the basis of (1.7) we find expressions for the stresses acting in the jet subjected to uniaxial tension in the course of bending:

$$\sigma_{nn} = \sigma_{bb} = -(\alpha/a)(1-ky), \tag{1.9}$$
$$\sigma_{\tau\tau} = -(\alpha/a)(1-ky) + 3^{(n+1)/2}\,K|D_{\tau\tau}|^n \operatorname{sgn} D_{\tau\tau}.$$

The latter expressions in conjunction with (1.1) and (1.5)—(1.7) enable us to calculate in the long-wave approximation the work L_1 of the internal forces in the designated element of the jet in time dt:

$$L_1 = \left\{\int_0^{\pi a_0/\chi}\left[\int_D(\sigma_{\tau\tau}D_{\tau\tau} - \sigma_{nn}D_{\tau\tau}/2 - \sigma_{bb}D_{\tau\tau}/2)(1-ky)\,dS\right]\lambda ds\right\}dt = 3^{\frac{n+1}{2}}K\frac{2a_0}{\chi}\left(A'\frac{\chi^2}{a_0^2}\right)^{n+1}\int_D|y|^{n+1}F(\varepsilon,n)\,dS\,dt, \tag{1.10}$$

$$F(\varepsilon,n) = \int_0^{\pi/2}|\sin x + \varepsilon\cos^2 x|^{n+1}dx,$$

where dS is an element of area of the jet cross section D and $\varepsilon = -A/y$.

The balance of energy indicates that the work of the distributed aerodynamic force is equal to the sum of the increments of the kinetic and surface energies of the jet and the work of the internal forces. We emphasize that in the energy balance the rheological properties of the liquid govern only the work of the internal forces.

The work of the distributed aerodynamic force and the increments of the kinetic and surface energies of the jet are readily calculated in the long-wave approximation by means of relations (1.1)—(1.4). In the case of small bending disturbances ($|\varepsilon| \sim A/a_0 \ll 1$) the work L_1 is determined mainly by the work of the moment of the internal stresses in rotation of the jet cross section, while the work of the longitudinal force in the cross section during elongation of the jet axis yields only a small correction. Accordingly, we obtain the following asymptotic representation for $F(\varepsilon, n)$:

$$F(\varepsilon,n) = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+2}{2}\right)\Big/\Gamma\left(\frac{n+3}{2}\right) + \varepsilon(n+1)\frac{\sqrt{\pi}}{4}\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+4}{2}\right) + \varepsilon^2\frac{3\sqrt{\pi}}{16}(n+1)\,n\,\Gamma\left(\frac{n}{2}\right)\Big/\Gamma\left(\frac{n+5}{2}\right). \tag{1.11}$$

Forming the energy balance and taking (1.11) into account, we carry out the integration over the cross section of the jet in (1.10) and arrive at an equation for the amplitude of the bending disturbance A(t):

$$A'' + 4\cdot 3^{\frac{n+1}{2}}\frac{K}{\rho\pi a_0^{n+1}}\chi^{2n+2}\left[\Gamma^2\left(\frac{n+2}{2}\right)\Big/\Gamma^2\left(\frac{n+3}{2}\right)\right]\frac{1}{n+3}(A')^n +$$

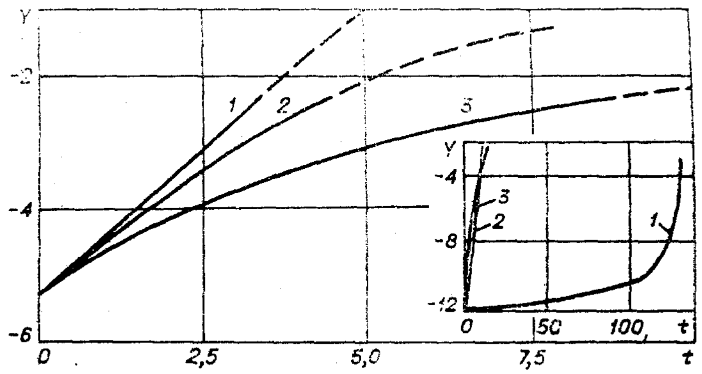

Fig. 1

$$+\frac{3}{2}3^{\frac{n-1}{2}}\frac{K}{\rho\pi a_0^{n+3}}\chi^{2n+2}\left\{\Gamma^2\left(\frac{n}{2}\right)\Big/\left[\Gamma\left(\frac{n+5}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\right]\right\}nA^2(A')^n$$
$$+A\chi^2\left(\frac{\alpha}{\rho a_0^3}-\frac{\rho_1 U_0^2}{\rho a_0^2}\right)=0. \quad (1.12)$$

Here the prime signifies differentiation with respect to t; the second term on the left describes the influence of the moment of the stress in the jet cross section in bending, and the third term describes the influence of the longitudinal force in the cross section on the bending process. For $n = 1$ and $K = \mu$ Eq. (1.12) goes over to the equation obtained in [1] for the amplitude of the transverse disturbances of a Newtonian fluid jet:

$$A''+\frac{3}{4}\frac{\mu}{\rho a_0^2}\chi^4 A'+\frac{9}{4}\frac{\mu}{\rho a_0^4}\chi^4 A^2 A'+A\chi^2\left(\frac{\alpha}{\rho a_0^3}-\frac{\rho_1 U_0^2}{\rho a_0^2}\right)=0. \quad (1.13)$$

An equation for A(t) can be formulated analogously in the case $a_0 \ll A \ll l$, i.e., $|\varepsilon| \gg 1$, when L_1 is determined mainly by the work of the longitudinal force in the cross section during elongation of the jet in the bending process, while the work of the internal stress moment provides only a correction. To calculate the work of the internal forces in this case it is necessary to formulate an asymptotic representation of the integral $\Phi(\varepsilon, n)=\int_0^{\pi/2}\left|\cos^2 x+\frac{1}{\varepsilon}\sin x\right|^{n+1}dx$ (it is seen at once that it will include 1, $1/\varepsilon$,

$$1/|\varepsilon|^{n+3/2},\ 1/\varepsilon^2 \text{ for } 0<n<1/2;\ 1,\ 1/\varepsilon,\ (1/\varepsilon^2)\ln(1/|\varepsilon|),$$
$$1/\varepsilon^2 \text{ for } n=1/2;\ 1,\ 1/\varepsilon,\ 1/\varepsilon^2 \text{ for } n>1/2).$$

For the numerical integration of Eq. (1.12) we use the following values of the parameters: $K = 10$ g/cm·sec^{2-n}, $\rho = 1$ g/cm^3, $a_0 = 10^{-1}$ cm, $\rho_1 = 10^{-3}$ g/cm^3, $U_0 = 10^3$ cm/sec. The surface tension of the liquid is disregarded; this assumption is not essential. The dimensionless wave number χ of the disturbance is taken equal to 0.667, which for the selected values of the parameters corresponds to the wavelength of an ultimately rapidly-growing small transverse disturbance of a Newtonian fluid jet $\left(\chi=\chi_*=\left[\frac{8}{9}\frac{\rho a_0^2}{\mu^2}\left(\rho_1 U_0^2-\frac{\alpha}{a_0}\right)\right]^{1/6}\right.$ see (1.13) [1, 2]). The results are represented in dimensionless form by the solid curves in Fig. 1. The disturbance amplitude $H_{max} = A$ is referred to its wavelength $l = 0.943$ cm, and the time to the characteristic rise time $T = 0.0047$ sec of small bending disturbances of a Newtonian fluid jet with the selected parameters $\left(T=\left(\frac{\rho\mu a_0^2}{\rho_1^2 U_0^4}\right)^{1/3}\right.$, see (1.13) and [1, 2]); $Y = \ln(H_{max})$.

The calculations according to Eq. (1.12) are carried out up to a value of the disturbance amplitude equal to the radius of the jet; for the selected values of the parameters this corresponds to $Y = -2.24$. We investigate cases in which at $t = 0$ the initial disturbances have, in dimensionless form, amplitudes $A = A' = A_0 = 5\cdot 10^{-3}$ and $A_0 = 5\cdot 10^{-6}$ (the results corresponding to the latter value of A_0 are shown in the inset in the lower right-hand corner of the figure). The curves are numbered as follows: 1) n = 0.5; 2) n = 1; 3) n = 1.5; these values span the cases of pseudoplastic, Newtonian, and dilatant fluids.

The initial disturbance amplitude $A_0 = 5\cdot10^{-3}$ and hence, the initial strain rate are sufficient for the effective viscosity of a pseudoplastic fluid jet not to be too great at the initial times: $\mu_1 = K(D_{\tau\tau})^{n-1} \simeq K\left(A'\frac{\chi^2}{a_0}\right)^{n-1} = 4.74$ P. This case corresponds to the rapid growth of bending disturbances in comparison with Newtonian ($\mu = 10$ P) and dilatant ($\mu_1 = 21.1$ P) fluid jets.

Actually, the diminution of viscous effects in connection with pseudoplastic fluid behavior brings us closer to a pure inertial solution of Eq. (1.12): $A \approx A_0 \exp(\gamma t)$, $\gamma = \chi \cdot \sqrt{\frac{\rho_1 U_0^2 T^2}{\rho a_0^2}} = 0.99$, $Y = -5.3 + 0.99t$ (in dimensionless form).

The growth of the exponent n in the rheological relation for a nonlinear viscous liquid in the case $A_0 = 5\cdot10^{-3}$ leads to the enhancement of viscous effects which stabilize the process. For example, in the case of small bending disturbances of a Newtonian fluid jet the dimensionless growth rate, according to (1.13), is equal to 0.69, which is the dimensionless coordinates of the figure in the case $A_0 = 5\cdot10^{-3}$ corresponds to the straight line $Y = -5.3 + 0.69t$. Finite bending disturbances of a Newtonian fluid jet grow even more slowly (curve 2 in Fig. 1) as a result of the stabilizing influence of the viscous stresses induced by elongation of the jet axis in bending (third term on the left-hand side in (1.12). In a dilatant fluid this nonlinear effect is enhanced by the growth of the viscosity with the strain rate (curve 3 in Fig. 1), further slowing the growth of the bending disturbances of the jet. Moreover, with an increase in n, the stabilizing influence of the viscous stresses associated with elongation of the jet axis sets in earlier, at smaller amplitudes of the disturbance waves.

For a very small initial disturbance ($A_0 = 5\cdot10^{-6}$) the strain rates are so small at the initial times, and the effective viscosity of the pseudoplastic fluid is so great ($\mu_1 = 150$ P), that the second term on the left-hand side in (1.12) becomes dominant and the growth of the bending disturbances is slowed down abruptly (see the inset in Fig. 1). In this case, increasing n tends to suppress the stabilizing viscous effects, as illustrated by the inset in Fig. 1 (for a Newtonian fluid $\mu = 10$ P, and for a dilatant $\mu_1 = 0.669$ P).

2. The analysis of the process of growth of long-wave bending disturbances of nonlinear viscous liquid jets for values of the amplitude $A > a_0$ can be continued with the application of the appropriate asymptotic representation for the case $|\varepsilon| \gg 1$. It is expected, however, that for sufficiently large disturbances nonlinear effects will produce considerable distortion of the jet axis and its single-harmonic representation (1.1) used in the energy method will yield substantial error. For the calculations in this case, therefore, it is logical to use the system of asymptotic quasi-one-dimensional equations for the dynamics of liquid jets [1, 3], which is easily generalized to the case of nonlinear viscous liquids. All that is required is to replace the expressions for the longitudinal force P and the stress moment **M** in the jet cross section; all other relations and equations remain unchanged.

Using the rheological relation (1.8), the expressions for the components of the strain-rate tensor, and the estimates for the stresses in the jet cross section [1, 3], we obtain linear expressions for the stresses and calculate the force P and the projections of **M** onto the normal, binormal, and tangent to the jet axis in the general case of nonplanar bending:

$$P = \left[3^{\frac{n+1}{2}} K\,|\lambda^{-1}V_{\tau,s} - kV_n|^n \operatorname{sgn}(\lambda^{-1}V_{\tau,s} - kV_n) - \alpha G\right] f + P_\alpha,$$

$$P_\alpha = 2\pi a\alpha\,(1 + \lambda^{-2}a_{,s}^2)^{-1/2},$$

$$M_n = 3^{\frac{n+1}{2}} Kn\,|\lambda^{-1}V_{\tau,s} - kV_n|^{n-1} I\,(\lambda^{-1}\Omega_{n,s} + k\Omega_\tau - \varkappa\Omega_b), \tag{2.1}$$

$$M_b = 3^{\frac{n+1}{2}} Kn|\lambda^{-1}V_{\tau,s} - kV_n|^{n-1} I\left(\lambda^{-1}\Omega_{b,s} + \varkappa\Omega_n - \frac{3}{2}k\lambda^{-1}V_{\tau,s} + \frac{3}{2}k^2V_n\right) - \alpha k I a^{-1}(1 + \lambda^{-2}a_{,s}^2)^{-3/2},$$

$$M_\tau = 3^{\frac{n-1}{2}} K\,|\lambda^{-1}V_{\tau,s} - kV_n|^{n-1} I\,(2\lambda^{-1}\Omega_{\tau,s} + k\lambda^{-1}V_{b,s} + k\varkappa V_n - k\Omega_n),$$

where **V** and **Ω** are the velocity of the center of the liquid cross section of the jet and the angular velocity of rotation of that cross section, $\varkappa$ is the torsion of the jet axis, G is the double mean curvature of the jet surface, $I = \pi a^4/4$ is the moment of inertia of the jet cross section, and λ is the elongation of the jet axis in nonplanar bending.

Relations (2.1) have been obtained on the assumption that the velocity of rotation of the liquid cross section is small in comparison with the strain rate of that cross section, corresponding either to large amplitudes of the bending disturbances ($|\varepsilon| \gg 1$) or to the absence of bending, as occurs in the case of capillary decay of rectilinear jets. In the latter case $\mathbf{\Omega} = 0$, k = $\varkappa$ = 0, λ = 1, and from (2.1) we obtain the expression used in [5, 6] for the longitudinal force in the cross section in calculations based on the quasi-one-dimensional equations for the capillary decay of power-law fluid jets.

We note that within the context of the quasi-one-dimensional approach it is also possible to obtain expressions for P and **M** corresponding to the case of very small bending disturbances ($|\varepsilon| \ll 1$) investigated here by the energy method. Of course, these expressions will differ from (2.1), and all other relations and equations of the quasi-one-dimensional theory [1, 3] will remain valid.

The computational algorithm in this case differs from the one used in the calculations of Newtonian fluid jets [1] only in the nonlinearity iterations associated with the double-sweep (modified Gaussian elimination) process.

The results of a numerical solution of the quasi-one-dimensional equations for the dynamics of liquid jets, continuing curves 1-3 of Fig. 1 into the domain of values Y > −2.24 ($A > \alpha_0$), are represented by the dashed curves. The solutions represented by the solid and dashed curves in Fig. 1 match smoothly. This indicates that the approximate energy method and the quasi-one-dimensional equations give consistent results.

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